The Tarski Laplacian

Hans Riess
University of Pennsylvania
hmr@seas.upenn.edu

Joint work with Paige Randall-North, Robert Ghrist,
Alejandro Parada-Mayorga, & Alejandro Ribeiro

We would like to thank our sponsors: the National Science Foundation & the Simons Foundation under the Mathematics of Deep Learning program, through the THEORINET (Transferable, Hierarchial Expressive, Optimal, Robust, Interpretable NETworks) grant (R. Vidal, PI).

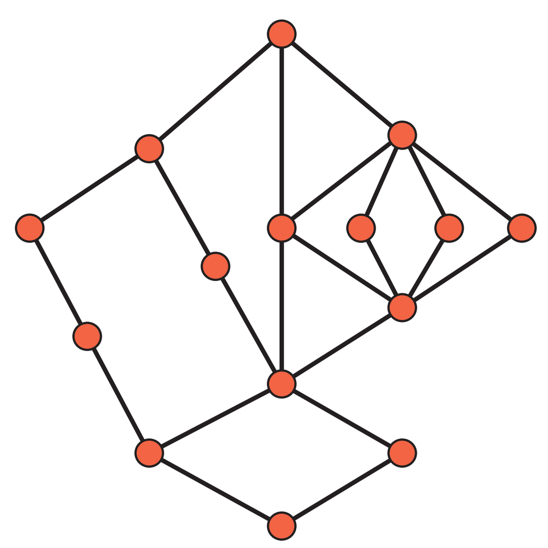
Lattices and Galois Connections

Lattices are algebraic structures–much like vector spaces or symmetry groups–that distinguish themselves by their hierarchical nature. Give any two lattice elements, one may perform lattice operations “meet” and “join” denoted $x \wedge y$ and $x \vee y$. These operations behave like intersection and union, satisfying a number of axioms (symmetry, idempotence, absorption, identity), and inherit an order $\leqslant$. Lattices also are assumed to have a top element 1 and a bottom element 0. Examples of lattices include the powerset 2^X with intersection and union, the unit interval $[0, 1]$ with minimum and maximum, and the collection of subspaces of a vector space with intersection and sum.

Lattices are related by structure-preserving maps between then. Theses are called Galois connections. Galois connection are to lattices as linear maps and their adjoints (transpose) are to inner-product spaces. Formally, given a lattice $\mathbf{P}$ and a lattice $\mathbf{Q}$, a Galois connection consists of a pair of maps $L\colon \mathbf{P} \leftrightarrows \mathbf{Q}\colon R$ such that $L(x) \geqslant y$ if and only if $R(y) \geqslant x$.

Example. Let X (e.g. users) and Y (e.g. movies) be sets, and $I \subseteq X \times Y$ a binary relation (e.g. user x watched movie y). Then, $(-)^{\uparrow}\colon 2^X \leftrightarrows 2^Y\colon (-)^{\downarrow}$ is a Galois connection given by

$$\sigma^{\uparrow} = \{y \in Y\colon (x, y) \in I \ \forall x \in X\};\ \tau^{\downarrow} = \{x \in X\colon (x, y) \in I \ y \in Y\}$$



	Space Wars	Lord of the Bands	Harry Ceramics
Alice			
Bob			
Eve			

Weighted Galois Connections

One challenge to designing a information processing systems with lattices is that there are no “coefficients,” a necessity in order to employ traditional training techniques in machine learning. This leads to a notion of weighted lattices. A weighted lattice is a collection of maps $[X, \mathbb{L}]$ for which X is a set and $\mathbb{L}$ is a lattice with additional structure called a residuated lattice. A residuated lattice is a tuple $\mathbb{L} = (L, \wedge, \vee, 0, 1, \otimes, e, \Rightarrow)$ such that $(L, \wedge, \vee, 0, 1)$ is a lattice and $(L, \otimes, e)$ is a monoid (set with associative binary operation & identity) with a notion of implication $\Rightarrow$ dual to $\otimes$. A notion of subsethood [Belohlavek 1999] quantifies the degree to which a two elements σ, τ of W “are related”

$$\mathcal{S}ub(\sigma, \tau) = \bigwedge_{x \in X} \sigma(x) \Rightarrow \tau(x)$$

Given a weighted relation $I\colon X \times Y \to \mathbb{L}$, there is a weighted Galois connection between $[X, \mathbb{L}]$ and $[Y, \mathbb{L}]$ given by

$$\sigma^{\uparrow}(y) = \bigwedge_{x \in X} \sigma(x) \Rightarrow I(x,y);\ \tau^{\downarrow}(y) = \bigwedge_{y \in X} \sigma(y) \Rightarrow I(x,y).$$

Such a Galois connection satisfies the property

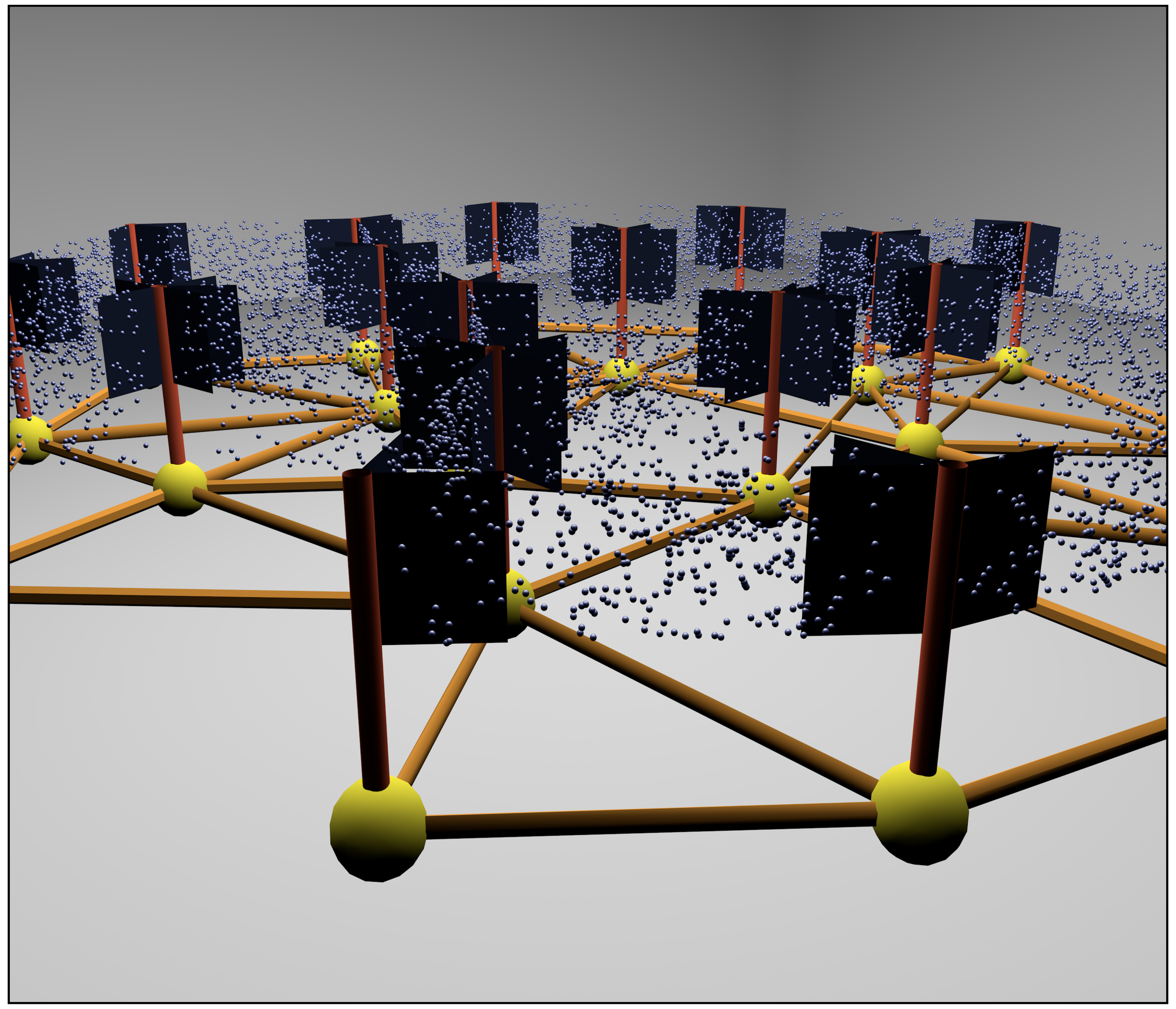
$$\mathcal{S}ub(\tau^{\downarrow}, \sigma) = \mathcal{S}ub(\sigma^{\uparrow}, \tau).$$

What is a Laplacian?

What is a Laplacian? In calculus, a Laplacian is a sum of 2nd order partial derivatives. In differential topology, a Laplacian is the Hodge Laplacian of a cochain complex. In graph signal processing (GSP), a Laplacian is a matrix with the same sparsity pattern as the graph. Maybe a Laplacian is best defined as an

operator driving harmonic dynamics via message-passing.

Harmony can differ depending on the context: in GSP Laplacian dynamics lead to eventual consensus across the nodes of the network. In this poster, we define a Laplacian called the Tarski Laplacian which acts on assignments of lattice-valued data to a network in order to reach two classes of harmonies: sections & concepts. Applications to GNNs are proposed.



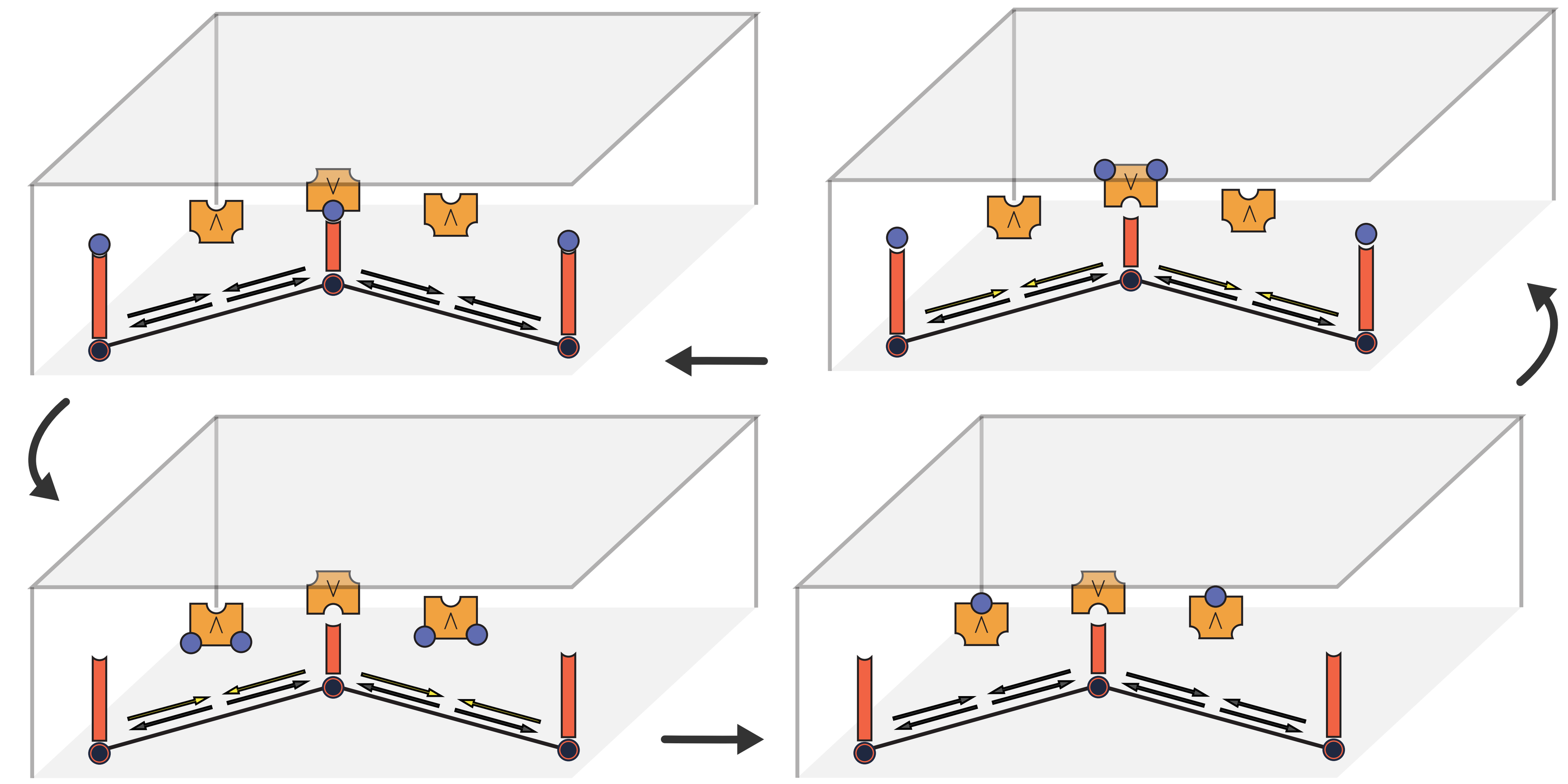
Sections vs. Concepts

Both sections and concepts are consistent assignments of lattice-valued data to a network. The nomenclature of a section comes from sheaf theory, a deep area of mathematics studying local-to-global behavior, where the term (formal) concept was coined by Wille (1982) in deference to ontology. While there are multiple ways to associate lattices to a network, we will narrow our focus to relation-weighted graphs. Let $\mathcal{G} = (\mathcal{V}, \mathcal{E}, X, Y, \mathbb{L}, \mathcal{I})$ be a graph with a collection

$$\mathcal{I} = \{I_{v,w}\colon X \times Y \to \mathbb{L}\}_{vw \in \mathcal{E}}$$

of relations. A section, then, is a collection $(\sigma_v\colon X \to \mathbb{L})_{v \in \mathcal{V}}$ such that $\sigma_v^{\uparrow e} = \sigma_w^{\uparrow e}$ for all (v, e, w).

A concept is a section with $(\tau_e\colon Y \to \mathbb{L})_{e \in \mathcal{E}}$ such that $\tau_e^{\downarrow v} = \tau_{e'}^{\downarrow v}$ for all (e, v, e').



Tarski Laplacian

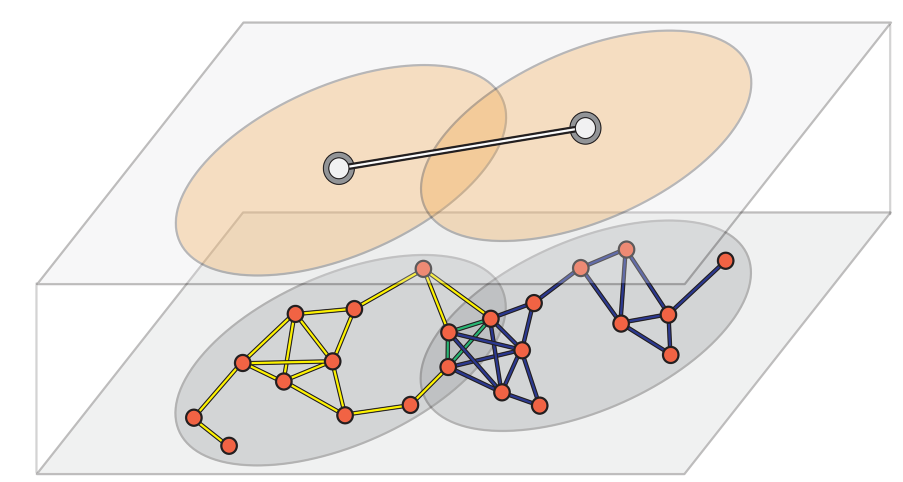
We are ready to define the Tarski Laplacian which acts on assignments of lattice-valued data over a network. The one direction of our research is using the Tarski Laplacian as a shift operator for GSP. Here’s the setup: let $\mathcal{G} = (\mathcal{V}, \mathcal{E}, X, Y, \mathbb{L}, \mathcal{I})$ be a relation-weighted graph. Then the Tarski Laplacian is the operator

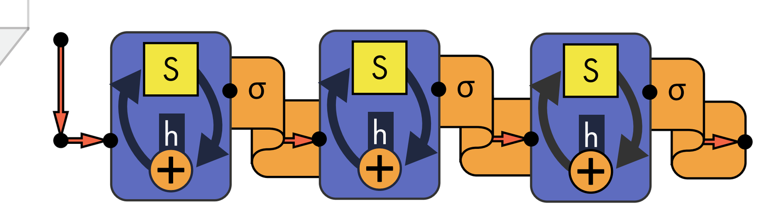
$$\mathrm{L}\colon \prod_{v \in \mathcal{V}} [X, \mathbb{L}] \to \prod_{v \in \mathcal{V}} [X, \mathbb{L}]$$

given by

$$(L\boldsymbol{\sigma})_v = \bigvee_{(w,e) \in \mathcal{N}(v)} (\sigma_v^{\uparrow e} \wedge \sigma_w^{\uparrow e})^{\downarrow v}$$

Theorem. Let $\rho \in \mathbb{L}$, then $\bigwedge_{v \in \mathcal{V}} \mathcal{S}ub(\sigma_v, (L\boldsymbol{\sigma})_v) \geqslant \rho$ if and only if $\mathcal{S}ub(\sigma_v^{\uparrow e}, \sigma_w^{\uparrow e}) \geqslant \rho$ for all $(v, e\ w)$. This theorem says the Tarski Laplacian, much like the graph Laplacian, “smoothes” out data in order to reach a section. (The figure above illustrates what is happening.)





Example. Let $X = \mathcal{V}_1$, $Y = \mathcal{V}_2$ be the vertex sets of two particular communication subgraphs $\mathcal{G}_1 = (\mathcal{V}_1, \mathcal{E}_1)$, $\mathcal{G}_2 = (\mathcal{V}_2, \mathcal{E}_2)$ in a sensor network. Let $\mathbb{L} = ([0, 1], \min, \max, 0, 1, \cdot, \to)$, a residuated lattice with $s \to t = \max(t/s, 1)$. Assume $\mathcal{V}_1 \cap \mathcal{V}_2 \neq \varnothing$. Then, $\mathcal{G}_1$ and $\mathcal{G}_2$ are connected (though possibly “distant”). We build a relation weighted graph with $I(v_1, w) = e^{-d(v1, w)}$ and $I'(v_1, w) = e^{-d(v1, w)}$ with $w \in \mathcal{V}_1 \cap \mathcal{V}_2$. With subgraph signals $\boldsymbol{s}_1\colon \mathcal{V}_1 \to [0,1]$, $\boldsymbol{s}_2\colon \mathcal{V}_2 \to [0,1]$, we can “harmonize” them by applying the weighted Tarski Laplacian

$$L(\boldsymbol{s}_1, \boldsymbol{s}_2) = \left((\boldsymbol{s}_1^{\uparrow} \wedge \boldsymbol{s}_2^{\uparrow})^{\downarrow 1}, (\boldsymbol{s}_1^{\uparrow} \wedge \boldsymbol{s}_2^{\uparrow})^{\downarrow 2}\right).$$

Explicitly, for $i \in \{1, 2\}$,

$$L(\boldsymbol{s}_1, \boldsymbol{s}_2)_i(v_i) = \min_{w \in \mathcal{V}_1 \cap \mathcal{V}_2} \frac{\min\{\min_{v_1 \in \mathcal{V}_1} \max(\frac{e^{-d(v_1,w)}}{\boldsymbol{s}_1(v_1)}, 1), \min_{v_2 \in \mathcal{V}_2} \max(\frac{e^{-d(v_2,w)}}{\boldsymbol{s}_2(v_2)}, 1)\}}{e^{-d(v_i,w)}}$$

We hypothesize that the weighted Tarski Laplacian prove useful in distributed training of GNNs.